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THE LEVI–LAX CONDITION FOR PARTIAL DIFFERENTIAL EQUATIONS WITH REAL CHARACTERISTICS OF CONSTANT MULTIPLICITY

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ABSTRACT

The object of this paper is to show that the Levi–Lax condition is necessary and sufficient for the Cauchy problem to be well-posed.

Introduction

We deal in this paper with conditions on partial differential equations with characteristics of constant multiplicity under which the Cauchy problem is well posed. P.D. Lax [9] and S. Mizohata [12] showed that the Cauchy problem cannot be well-posed unless the characteristics are all real. If, in addition, these roots are simple (their multiplicity is one), the well-posedness of the Cauchy problem was proved by I.G. Petrowsky [15], J. Leray [10] and L. Gårding [3]. For equations in two independent variables having characteristics of multiplicity higher than one, the Cauchy problem was proved to be well-posed under a condition on the lower-order terms by E.E. Levi [11] and A. Lax [8]. This condition has come to be known as the Levi–Lax condition. It was extended for equations having several independent variables, but whose characteristics are of multiplicity at most two, by M. Yamaguti [18]. Later, S. Mizohata and and Y. Ohya [13], [14] formulated an alternative condition similar in form to the Levi–Lax condition (also for several variables and multiplicity at most two) and showed the condition to be necessary as well as sufficient in their restricted case. H. Flaschka and G. Strang [2] formulated still another condition without any restrictions on the size of the multiplicity or the number of variables and showed

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it to be necessary. J. Chazarain [1] then showed that the Flaschka–Strang form of the condition was sufficient as well.

In this paper we will give first a new proof of the theorem proved originally by Petrowsky for what are now called strictly hyperbolic operators. Next, we will extend and prove sufficient the Levi–Lax condition for equations unrestricted either in the number of independent variables or in the size of the multiplicity of the characteristics. In doing so, we will show how Hörmander's domination relation (namely that P, the operator under consideration, is weaker than P_m, its principal part: see L. Hörmander [4]) extends for operators having variable coefficients. We will also formulate an extension of the Mizohata–Ohya condition to equations having characteristics of arbitrary multiplicity. We will then show the equivalence of all of these conditions (the Levi–Lax condition, the generalized Mizohata–Ohya condition and the Flaschka–Strang condition), thus proving, among other things, that the Levi–Lax condition is necessary as well as sufficient in order for the Cauchy problem to be well-posed.

1. Statement of problem and notation

First, recall the problem. Let

$$P(x,t,D_x,D_t) = P_m(x,t,D_x,D_t) + P_{m-1}(x,t,D_x,D_t) + \cdots$$

be a linear partial differential operator of order m and the P_i are homogeneous of order i in (x,t): $x = (x_1,\cdots,x_n) \in \mathbf{R}^n$, $t \in \mathbf{R}^1$.

Let $P_m(x,t,\xi,\tau)$ be the leading symbol of P where $\xi = (\xi_1,\cdots,\xi_n) \in \mathbf{R}^n$ and $\tau \in \mathbf{R}^1$.

Assume the hyperplane $t = 0$ is non-characteristic at the origin with respect to P; i.e., $P_m(0,0,0,1) \neq 0$. The Cauchy problem is to find a solution v of $Pv = f$ in a neighborhood of the origin with given (say homogeneous) Cauchy data on the plane

$$t = 0: D_t^j v\,\big|_{t=0} = 0, \qquad j = 0,\cdots,m-1.$$

The Cauchy problem is said to be well-posed (in the sense of Hadamard) if a solution exists, is unique, and depends smoothly on the initial data and the function f.

DEFINITION. We will call an operator P hyperbolic if its associated C^∞ Cauchy problem is well-posed.

Since $t = 0$ is non-characteristic at the origin with respect to P we may assume that the coefficient of D_t^m in P_m is 1.

For an n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ of non-negative integers, we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

$$\partial_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right), \qquad \partial_t = \frac{\partial}{\partial t},$$

$$D_x = \frac{1}{i}\partial_x, \qquad D_t = \frac{1}{i}\partial_t.$$

$$D^r = \sum_{|\alpha|=r} D^\alpha.$$

L_x^γ denotes the class of homogeneous pseudo-differential operators of order γ in the x-variables and S_x^γ is its corresponding symbol space. See J.J. Kohn and L. Nirenberg [7] for more details. $L_{x,t}^\gamma$ is the class of homogeneous operators differential in t and pseudo-differential in x of order γ in (x, t). $S_{x,t}^\gamma$ is its symbol space. (u, v) is the L_2 scalar product of u and v with respect to the x-variables. $\| u \|$ is the corresponding L_2 norm of u. H_s is the Hilbert space with norm $\| u(\cdot, t) \|_s$ defined by

$$\| u \|_s^2 = \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi$$

where $\hat{u}$ is the Fourier transform of $u(x, t)$ with respect to the x-variable;

$$[A, B] = AB - BA.$$

Finally, C will be used to denote any constant and may be varied from line to line.

2. Conditions and statements of theorems

The first condition restricts the type of characteristics we may allow.

Proposition 2.1. (P.D. Lax [9], S. Mizohata [12]). *Let $t = 0$ be non-characteristic with respect to the operator P. If the Cauchy problem is well-posed, then the roots τ of $P_m(x, t, \xi, \tau) = 0$ are all real.*

Proof. See V. Ya. Ivrii and V. M. Petkov [6] or L. Hörmander [5] for a particularly simple proof.

If the characteristics of P are, in addition, simple (the multiplicity is one) the Cauchy problem is well-posed.

THEOREM 1 (I. G. Petrowsky [15]). *Suppose $t = 0$ is non-characteristic at the origin with respect to P. Suppose the characteristics are simple and real. Then the Cauchy problem is well-posed.*

We will give a new proof of Theorem 1 later. It will become useful in proving results for equations with characteristics having multiplicities of higher order.

Among partial differential equations with characteristics of higher multiplicity, we will consider only those for which the multiplicity is constant. This means that if τ_1 and τ_2 are distinct zeros of $P_m(x,t,\xi,\tau) = 0$ on $|\xi| = 1$, then $|\tau_1 - \tau_2| \geqq \varepsilon$ where ε is a fixed positive number independent of x, t and ξ.

REMARK. Results for equations whose characteristics have variable multiplicity have been given by Hörmander [5], Ivrii and Petkov [6] and Zeman [19].

If P has characteristics of constant multiplicity, we can write $P_m(x,t,\xi,\tau)$ in the form

$$P_m(x,t,\xi,\tau) = \prod_{i=1}^{p} (\tau - \lambda_i(x,t,\xi))^{r_i}, \qquad \lambda_i(x,t,\xi) \in S^1_x,$$

where $|\lambda_i(x,t,\xi) - \lambda_j(x,t,\xi)| \geqq \varepsilon$ for $(x,t) \in \Omega$ and $|\xi| = 1$, and where $\Omega = \{(x,t): |x| \leqq \bar{r},\ 0 \leqq t \leqq T\}$.

Let $\partial_i = D_t - \lambda_i(x,t,D_x)$, where

$$\lambda_i(x,t,D_x)\, u(x,t) = \left(\frac{1}{2\pi}\right)^{n/2} \int e^{ix\xi} \lambda_i(x,t,\xi) \hat{u}(\xi,t) d\xi.$$

Suppose P_m is the leading part of $\Pi_m = \Pi_{i=1}^{p} \partial_i^{r_i}$. Rewrite $P = P_m + P_{m-1} + \cdots$ as follows: $P = \Pi_m + P'_{m-1} + P'_{m-2} + \cdots$, where P'_{m-j} is an operator of order $m-j$ (not necessarily homogeneous). We will now state the condition on the lower order terms P'_{m-j} which will turn out to be both necessary and sufficient for hyperbolicity.

Condition (L)

$$P'_{m-j}(x,t,D_x,D_t) = M_j(x,t,D_x,D_t) \prod_{i=1}^{p} \partial_i^{[r_i-j]}, \ j = 1,2,\cdots,r-1,$$

for some $M_j \in L^{\langle j \rangle}_{x,t}$, where $\langle j \rangle = m - j - \Sigma_{i=1}^{p}[r_i - j]$, and where $[r_i - j] = \max\{r_i - j, 0\} \cdot r = \max r_i$.

REMARK 1. At the points $\tau = \lambda_j(x,t,\xi)$, $r_j \geqq 2$, $P'_{m-1}(x,t,\xi,\tau)$ is equal to the subprincipal symbol of P given by

$$P^{s}_{m-1}(x,t,\xi,\tau)=P_{m-1}(x,t,\xi,\tau)-\frac{1}{2i}\left\{\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{\partial}{\partial \xi_i}P_m+\frac{\partial}{\partial t}\frac{\partial}{\partial \tau}P_m\right\},$$

which is invariant at these points.

REMARK 2. We shall now describe a way to determine whether an operator P satisfies condition (L). Let $P=P_m+P_{m-1}+P_{m-2}+\cdots$. Replacing P_m by Π_m we introduce an error term which we have to account for. Hence $P=\Pi_m+(P_m-\Pi_m+P_{m-1})+\cdots$. Let

$$\tilde{P}_{m-1}(x,t,D_x,D_t)=P_m(x,t,D_x,D_t)-\Pi_m+P_{m-1}(x,t,D_x,D_t).$$

If condition (L) is satisfied, we can factor $\tilde{P}_{m-1}(x,t,\xi,\tau)$, the principal symbol of $\tilde{P}_{m-1}$, as follows:

$$\tilde{P}_{m-1}(x,t,\xi,\tau)=M_1(x,t,\xi,\tau)\prod_{i=1}^{p}(\tau-\lambda_i(x,t,\xi))^{[r_i-1]}.$$

Finally,

$$P'_{m-1}(x,t,D_x,D_t)=M_1(x,t,D_x,D_t)\prod_{i=1}^{p}(D_t-\lambda_i(x,t,D_x))^{[r_i-1]}.$$

More generally,

$$\tilde{P}_{m-j}(x,t,D_x,D_t)=(\tilde{P}_{m-j+1}-P'_{m-j+1}+P_{m-j})(x,t,D_x,D_t).$$

If condition ($\tilde{\mathrm{L}}$) is satisfied we can factor $\tilde{P}_{m-j}(x,t,\xi,\tau)$, the principal symbol of $\tilde{P}_{m-j}$, as follows:

$$\tilde{P}_{m-j}(x,t,\xi,\tau)=M_j(x,t,\xi,\tau)\prod_{i=1}^{p}(\tau-\lambda_i(x,t,\xi))^{[r_i-j]}.$$

Then

$$P'_{m-j}(x,t,D_x,D_t)=M_j(x,t,D_x,D_t)\prod_{i=1}^{p}(D_t-\lambda_i(x,t,D_x))^{[r_i-j]}.$$

If the above operation fails at any step, then condition (L) is not satisfied.

For partial differential equations having multiple characteristics, we have the following theorem:

THEOREM 2. *Suppose $t=0$ is non-characteristic at the origin with respect to P (as described above). Then the Cauchy problem for operator P is well-posed if and only if condition* (L) *holds.*

REMARK. Condition (L) was first formulated by E. E. Levi [11] and then rediscovered by A. Lax [8]. Both dealt with equations in two independent variables. In that case $\lambda_i(x,t,D_x) = f_i(x,t)D_x$ for some real smooth function f_i. Similarly $M_j(x,t,D_x,D_t)$ is some *partial* differential operator in x and t. Otherwise condition (L) as given here is precisely the same as that of A. Lax.

3. Technical lemmas

Before we prove Theorems 1 and 2, we need some preliminary lemmas.

LEMMA 3.1. *If $s = \partial_1^{m_1}\partial_2^{m_2}\cdots\partial_k^{m_k}$ is of order m (i.e. $\sum_{i=1}^k m_i = m$) and $\bar{s}$ is a permutation of s, then*

$$s - \bar{s} = h_{m-1}(x,t,D_x,D_t) + h_{m-2}(x,t,D_x,D_t) + \cdots,$$

where $h_{m-j}(x,t,D_x,D_t)$ satisfies condition (L) (*by which we mean that it is of the same form as P'_{m-j} in condition* (L)).

PROOF. It suffices to carry out the proof for the special permutation

$$s = \partial_1^{m_1}\cdots\partial_i^{m_i-1}\partial_i\partial_j\partial_j^{m_j-1}\cdots\partial_k^{m_k},$$

$$\bar{s} = \partial_1^{m_1}\cdots\partial_i^{m_i-1}\partial_j\partial_i\partial_j^{m_j-1}\cdots\partial_k^{m_k};$$

$$s - \bar{s} = \partial_1^{m_1}\cdots\partial_i^{m_i-1}[\partial_i,\partial_j]\,\partial_j^{m_j-1}\partial_k^{m_k}.$$

Now $[\partial_i,\partial_j] = b_1(x,t,D_x) + N_1$, where $b \in L_x^1$ and $N_1 \in L_x^0$. Hence,

$$s - \bar{s} = \partial_1^{m_1}\cdots\partial_i^{m_i-1}b_1\partial_j^{m_j-1}\cdots\partial_k^{m_k} + \partial_1^{m_1}\cdots\partial_i^{m_i-1}N_1\partial_j^{m_j-1}\cdots\partial_k^{m_k}.$$

We will now show this is equal to

$$\sigma = b_1\partial_1^{m_1}\cdots\partial_i^{m_i-1}\partial_j^{m_j-1}\cdots\partial_k^{m_k} + T,$$

where T represents the lower order terms, all of the same form as h_{m-j}.

Now $[\partial_i, b_1] = b_2 + N_2$, where $b_2 \in L_x^1$ and $N_2 \in L_x^0$. Hence,

$$s - \bar{s} = \partial_1^{m_1}\cdots\partial_i^{m_i-2}b_1\partial_i\partial_j^{m_j-1}\cdots\partial_k^{m_k} + \partial_1^{m_1}\cdots\partial_i^{m_i-2}b_2\partial_j^{m_j-1}\cdots\partial_k^{m_k}$$
$$+ \partial_1^{m_1}\cdots\partial_i^{m_i-1}N_1\partial_j^{m_j-1}\cdots\partial_k^{m_k} + \partial_1^{m_1}\cdots\partial_i^{m_i-2}N_2\partial_j^{m_j-1}\cdots\partial_k^{m_k}.$$

These terms are of the same form as before except that the first term b_1 has moved leftward and the second and fourth terms contain one less ∂_i than $s - \bar{s}$ did before, and so after a number of steps it is clear that

$$s - \bar{s} = b_1\partial_1^{m_1}\cdots\partial_i^{m_i-1}\partial_j^{m_j-1}\cdots\partial_k^{m_k} + N_1\partial_1^{m_1}\cdots\partial_i^{m_i-1}\partial_j^{m_j-1}\cdots\partial_k^{m_k}$$
$$+ b_2\partial_1^{m_1}\cdots\partial_i^{m_i-2}\partial_j^{m_j-1}\cdots\partial_k^{m_k} + \cdots$$
$$= h_{m-1} + h_{m-2} + \cdots,$$

where h_{m-1} is the first term and h_{m-2} is the sum of the second and third terms, and so on. It is evident that the h_{m-j} are of the form required.

REMARK. This lemma shows that condition (L) is invariant under an arbitrary permutation of the ∂_i.

LEMMA 3.2. *Suppose* $|\lambda_1(x,t,\xi)-\lambda_2(x,t,\xi)|\geqq \varepsilon$ *for some* $\varepsilon>0$, *for* $|\xi|=1$. *Then for any operator* $g(x,t,D_x,D_t)\in L^1_{x,t}$, *we can find* $c_1, c_2, N\in L^0_x$ *such that*

$$c_1\,\partial_1 + c_2\,\partial_2 = g(x,t,D_x,D_t) + N(x,t,D_x).$$

PROOF. g is of the form $g(x,t,D_x,D_t)=a(x,t,D_x)D_t - b(x,t,D_x)$, where $a\in L^0_x$ and $b\in L^1_x$. Hence, to find c_1 and c_2, we have to solve the following system of pseudo-differential equations (N will turn out to be some lower order term which appears when we solve the system):

$$c_1(D_t-\lambda_1(x,t,D_x)) + c_2(D_t-\lambda_2(x,t,D_x)) = a\,D_t - b.$$

This implies that

$$c_1(x,t,D_x) + c_2(x,t,D_x) = a(x,t,D_x)$$

and

$$c_1(x,t,D_x)\,\lambda_1(x,t,D_x) + c_2(x,t,D_x)\,\lambda_2(x,t,D_x) = b(x,t,D_x).$$

Using elliptic theory, we can solve this system for c_1 and c_2, modulo lower order terms which belong to L^0_x, since the matrix

$$\begin{pmatrix} 1 & 1 \\ \lambda_1(x,t,\xi) & \lambda_2(x,t,\xi) \end{pmatrix}$$

is non-singular for $|\xi|=1$; this completes the proof.

PROPOSITION 3.3. *Suppose* $\lambda_i(x,t,\xi)$ *is real for* $(x,t,\xi)\in\Omega\times\mathbf{R}^n\setminus\{0\}$, *where* $\Omega=\{(x,t): |x|\leqq \bar r,\ 0\leqq t\leqq T\}$. *For every real* s, *there exists a constant* $C(s)$ *independent of* u *such that*

$$\|u(\cdot,t)\|_s \leqq C(s)T\,\|\partial_i u(\cdot,t)\|_s \tag{3.1}$$

if $u(x,0)=0$, *where* $C(s)\ T<1$ *if* T *is small enough.*

PROOF.

$$\frac{d}{dt}(u,u)_s = 2\,\mathrm{Re}\,(u_t,u)_s. \tag{3.2}$$

Let $\partial_j u = D_t u - \lambda_j(x, t, D_x)u = v$. Then $D_t u = v + \lambda u$. Hence $u_t = iv + i\lambda u$, since $D_t = (1/i)\, d/dt$. Substituting into (3.2), we have

$$\frac{d}{dt}(u, u)_s = 2\,\mathrm{Re}\,(iv + i\lambda u, u)_s$$

$$= 2\,\mathrm{Re}\,(iv, u)_s + 2\,\mathrm{Re}\,(i\lambda u, u)_s\,;$$

$2\,\mathrm{Re}(i\lambda u, u)_s = (i[\lambda - \lambda^*]u, u)_s$, where λ^* is the formal adjoint of λ with respect to the inner product. Since the symbol of λ is real, λ differs from λ^* by an operator of order zero. Hence

$$(i[\lambda - \lambda^*]u, u)_s \leqq C_1 \| u \|_s^2.$$

This implies that

$$\frac{d}{dt}(u, u)_s \leqq 2\,\mathrm{Re}\,(iv, u)_s + C \| u \|_s^2.$$

Applying Cauchy's inequality,

$$2\,\mathrm{Re}\,(iv, u)_s \leqq C \| u \|_s \| v \|_s.$$

Hence

$$\frac{d}{dt}(u, u)_s \leqq C \| u \|_s^2 + C \| v \|_s \| u \|_s. \tag{3.3}$$

Multiplying (3.3) by $(T - t)$ and integrating, we have

$$\int_0^T \| u \|_s^2 dt \leqq CT \int_0^T \| u \|_s^2 dt + CT \left[\int_0^T \| u \|_s^2\, dt \right]^{1/2} \left[\int_0^T \| v \|_s^2\, dt \right]^{1/2} \tag{3.4}$$

from which it follows that for T small, $\| u \|_s \leqq CT \| v \|_s$, where C is a function of s alone. Hence, $\| u \|_s \leqq CT \| \partial_i u \|_s$, where $CT < 1$ for T small enough.

4. Proof of Theorem 1

The basic step in the proof of Theorem 1 is proving the following energy estimate.

PROPOSITION 4.1. *Let the hypotheses of Theorem 1 hold. Then there is a constant C independent of u such that for T and r sufficiently small, we have*

$$\frac{1}{T} \sum_{|\alpha| \leq m-1} \| D^\alpha u \|_s \leqq C \| Pu \|_s,$$

for $u \in C_0^\infty(\Omega)$, *where* $\Omega = \{(x,t): |x| \leqq r,\ 0 \leqq t \leqq T\}$, *if* $D_t^j u(x,t)|_{t=0} = 0$, *for* $j = 0, \cdots, m-1$.

REMARK. Proposition 4.1 remains true if P, which is a partial differential operator here (as in Theorem 1), is replaced by an operator, differential in t and pseudodifferential in x. It is with P in this form that Proposition 4.1 is used later in proving the sufficiency part of Theorem 2.

Before we prove Proposition 4.1, we first present two lemmas which will help simplify its proof.

LEMMA. 4.2. *Let* s, s' *be two real numbers such that* $s' < s$, $-n/2 \leqq s$. *Then to every* $\varepsilon > 0$ *there is a number* $\eta > 0$ *such that if the diameter of a compact set* K *is* $\leqq \eta$, *we have for all* $u \in H_s(K)$, $\| u \|_{s'} \leqq \varepsilon \| u \|_s$.

PROOF. See F. Trèves (theorem 0.41 in [17]).

COROLLARY 4.3. *For* r *sufficiently small, it suffices to prove in Proposition* 4.1 *that*

$$\frac{1}{T} \| D^{m-1} u \|_s \leqq C \| Pu \|_s.$$

LEMMA. 4.4. *Let* $R(x,t,D_x,D_t)$ *be an operator of order less than* m. *If Proposition* 4.1 *holds for* P, *it will still be true if* P *is replaced by* $P + R$.

PROOF. Since

$$\| Pu \|_s \leqq \| (P+R)u \|_s + \| Ru \|_s$$

$$\leqq \| (P+R)u \|_s + C \sum_{|\alpha| \leqq m-1} \| D^\alpha u \|_s,$$

then

$$\frac{1}{T} \sum_{|\alpha| \leqq m-1} \| D^\alpha u \|_s \leqq C \| (P+R)u \|_s + C \sum_{|\alpha| \leqq m-1} \| D^\alpha u \|_s. \tag{4.1}$$

Since $T < 1/C$ for T sufficiently small, we can absorb the second term of the right-hand side (r.h.s.) of (4.1) into the left-hand side (l.h.s.) and get

$$\frac{1}{T} \sum_{|\alpha| \leqq m-1} \| D^\alpha u \|_s \leqq C \| (P+R)u \|_s.$$

PROOF OF PROPOSITION 4.1. Invoking Corollary 4.3 and Lemma 4.4, it suffices to prove the following:

(4.2) $$\frac{1}{T}\| D^{m-1}u \|_s \leqq C \| \Pi_m \|_s,$$

where $\Pi_m = \partial_i \cdots \partial_m$, $\partial_i = D_t - \lambda_i(x,t,D_x)$, and $|\lambda_i(x,t,\xi) - \lambda_j(x,t,\xi)| \geqq \varepsilon$. If $m = 1$, (4.2) holds because of Proposition 3.3.

Now assume the theorem is true for $m \geqq 1$. We will prove that it is then true for $m+1$. We have, by the induction hypothesis,

(4.3) $$\frac{1}{T}\| D^{m-1}(\partial_{m+1}u)\|_s \leqq C \|(\partial_1 \cdots \partial_m)(\partial_{m+1}u)\|_s$$

and

(4.4) $$\frac{1}{T}\| D^{m-1}(\partial_m u)\|_s \leqq C \|(\partial_1 \cdots \partial_{m-1}\partial_{m+1})(\partial_m u)\|_s$$

since the operator $\partial_1 \cdots \partial_{m-1}\partial_{m+1}$ is of the same form as $\partial_1 \cdots \partial_m$. Now,

$$\partial_1 \cdots \partial_{m+1} = \partial_1 \cdots \partial_{m-1}\,\partial_{m+1}\,\partial_m + h,$$

where $h \in L^m_{x,t}$. Combining this with (4.4) we have

(4.5) $$\begin{aligned}\frac{1}{T}\| D^{m-1}(\partial_m u)\|_s &\leqq \| \partial_1 \cdots \partial_{m+1}u - hu \|_s \\ &\leqq C \| \partial_1 \cdots \partial_{m+1}u \|_s + C\| D^m u \|_s.\end{aligned}$$

Adding (4.3) and (4.5) yields

(4.6) $$\frac{1}{T}(\| D^{m-1}(\partial_m u)\|_s + \| D^{m-1}(\partial_{m+1}u)\|_s) \leqq C\| \partial_1 \cdots \partial_{m+1}u \|_s + C \| D^m u \|_s.$$

We will now show that (4.6) implies the following estimate:

(4.7) $$\frac{1}{T}(\| \partial_m(D^{m-1}u)\|_s + \| \partial_{m+1}(D^{m-1}u)\|_s) \leqq C \| \partial_1 \cdots \partial_{m+1}u \|_s + C \| D^m u \|_s.$$

Indeed, by Proposition 3.3,

(4.8) $$\frac{1}{T}\| D^{m-1}u \|_s \leqq C \| \partial_i D^{m-1}u \|_s.$$

Since $[D^{m-1}, \partial_i]$ is an operator of order $m-1$, we can show as before that

(4.9) $$\frac{1}{T}\| D^{m-1}u \|_s \leqq C \| D^{m-1}\partial_i u \|_s.$$

Hence combining (4.8) and (4.9) we have

$$2C\| D^{m-1}\partial_i u \|_s \geqq C\| \partial_i D^{m-1} u \|_s - C_2\| D^{m-1} u \|_s + \frac{1}{T}\| D^{m-1} u \|_s,$$

for some C_2. This implies that

$$2\,C\| D^{m-1}\partial_i u \|_s \geqq C\| \partial_i D^{m-1} u \|_s,$$

and (4.7) follows easily. (4.7) implies that

$$\frac{1}{T}\| (a\partial_m + b\partial_{m+1}) D^{m-1} u \|_s \leqq C\| \partial_1 \cdots \partial_{m+1} u \|_s,$$

for any $a, b \in L^0_x$. After applying Lemma 3.2, the following estimate follows easily:

$$\frac{1}{T}\| D^m u \|_s \leqq \frac{1}{T}\| D(D^{m-1}u)\|_s \leqq C\| \partial_1 \cdots \partial_{m+1} u \|_s.$$

Hence (4.2) holds for all m and the proof is complete.

Proof of Theorem 1. Proving that the Cauchy problem is well-posed by utilizing an energy estimate is standard. Among other places, a proof can be found in Zeman [19].

5. Sufficiency of condition (L)

We will show that condition (L) is sufficient for the Cauchy problem to be well-posed by proving an energy estimate. That the Cauchy problem is then well-posed follows as in the proof of Theorem 1.

Proposition 3. *Suppose $t = 0$ is non-characteristic with respect to P, a partial differential operator with characteristics of constant multiplicity. Suppose condition (L) holds. Then there is a constant C independent of u such that for T and $\bar{r}$ sufficiently small, we have*

$$\left(\frac{1}{T}\right)^r \sum_{|\alpha|\leqq m-r} \| D^\alpha u \|_s \leqq C\| Pu \|_s \tag{5.1}$$

for $u \in C_0^\infty(\Omega)$, where $\Omega = \{(x,t): |x| \leqq \bar{r}, 0 \leqq t \leqq T\}$, if $D_t^j u(x,t)\big|_{t=0} = 0$, for $j = 0, \cdots, m-1$. r represents here the maximum multiplicity of the characteristics of P.

Proof. Recall that $\Pi_m = \Pi_{i=1}^p \partial_i^{r_i}$, $\partial_i = D_t - \lambda_i(x,t,D_x)$, $r = \max r_i$, and $\Sigma_{i=r}^p r_i = m$ After a permutation, if necessary, we may assume that $r_1 \leqq r_2 \leqq \cdots \leqq r_p$. Let

$$\Pi_m' = \left(\prod_{i=1}^p \partial_{p-i+1}\right)^{r_1}\left(\prod_{i=1}^{p-1} \partial_{p-i+1}\right)^{r_2-r_1} \cdots \left(\prod_{i=1}^{p-k} \partial_{p-i+1}\right)^{r_{k+1}-r_k} \cdots (\partial_p)^{r_p-r_{p-1}}.$$

Then Π'_m is a product of r operators, each of the form $\Pi\partial_i$. Hence $\Pi'_m = P_{m_1}P_{m_2}\cdots P_{m_r}$, where each P_{m_i} is of the form $\Pi\partial_i$ of order m_i, with $m_1+m_2+\cdots+m_r = m$. More specifically, if $r_p = r_{p-1} = \cdots = r_{s+1} > r_s$, then, for instance, $P_{m_r} = \Pi_{i=1}^{p-s+1}\partial_{p-i+1}$. Note that each P_{m_i} satisfies the conditions of Theorem 1. Hence we can apply Proposition 4.1 to each of the P_{m_i}:

$$\frac{1}{T}\| D^{m_i-1}u \|_s \leqq C \| P_{m_i}u \|_s. \tag{5.2}$$

This implies that

$$\frac{1}{T}\| D^{m_1-1}P_{m_2}\cdots P_{m_r}u \|_s \leqq C \| P_{m_1}\cdots P_{m_r}u \|_s.$$

We will now commute D^{m_1-1} with P_{m_2}:

$$D^{m_1-1}P_{m_2}P_{m_3}\cdots P_{m_r} = P_{m_2}D^{m_1-1}P_{m_3}\cdots P_{m_r} + [D^{m_1-1}, P_{m_2}]P_{m_3}\cdots P_{m_r},$$

where $[D^{m_1-1}, P_{m_2}] = q$, an operator of order m_1+m_2-2. Applying Proposition 4.1 again, we have

$$\frac{1}{T}\| D^{m_1+m_2-2}P_{m_3}\cdots P_{m_r}u \|_s \leqq C \| P_{m_2}(D^{m_1-1}P_{m_3}\cdots P_{m_r}u) \|_s.$$

This implies that

$$\begin{aligned} \frac{1}{T}\| D^{m_1+m_2-2}P_{m_3}\cdots P_{m_r}u \|_s &\leqq C \| D^{m_1-1}P_{m_2}P_{m_3}\cdots P_{m_r}u - qP_{m_3}\cdots P_{m_r}u \|_s \\ &\leqq C \| D^{m_1-1}P_{m_2}\cdots P_{m_r}u \|_s + C \| D^{m_1+m_2-2}P_{m_3}\cdots P_{m_r}u \|_s. \end{aligned} \tag{5.3}$$

Since $1/T > C$ for T sufficiently small, we can absorb the second term of the r.h.s. of (5.3) into the l.h.s. and we have

$$\frac{1}{T}\| D^{m_1+m_2-2}P_{m_3}\cdots P_{m_r}u \|_s \leqq C \| D^{m_1-1}P_{m_2}P_{m_3}\cdots P_{m_r}u \|_s.$$

This implies that

$$\begin{aligned} \left(\frac{1}{T}\right)^2 \| D^{m_1+m_2-2}P_{m_3}\cdots P_{m_r}u \|_s &\leqq C_1\left(\frac{1}{T}\right) \| D^{m_1-1}P_{m_2}\cdots P_{m_r}u \|_s \\ &\leqq C \| P_{m_1}\cdots P_{m_r}u \|_s. \end{aligned} \tag{5.4}$$

We keep going in this manner, peeling off the P_{m_i} one by one until we exhaust them and get

$$(5.5)\qquad \sum_{j=1}^{r}\left(\frac{1}{T}\right)^{j}\left\|D^{m_1+\cdots+m_j-j}\prod_{i=j+1}^{r}P_{m_i}u\right\|_s \leqq C\|\Pi'_m u\|_s,$$

where the order of the operator in the l.h.s. of (5.5) is equal to $m_1+\cdots+m_j-j+(m_{j+1}+\cdots+m_r)=m-j$. Set $t_j=m_1+\cdots+m_j-j$. Then (5.5) becomes

$$(5.6)\qquad \sum_{j=1}^{r}\left(\frac{1}{T}\right)^{j}\left\|D^{t_j}\prod_{i=j+1}^{r}P_{m_i}u\right\|_s \leqq C\|\Pi'_m u\|_s.$$

We will show that (5.6) implies that

$$(5.7)\qquad \sum_{j=1}^{r}\left(\frac{1}{T}\right)^{j}\left\|D^{t_j}\prod_{i=1}^{p}\partial_i^{[r_i-j]}u\right\|_s \leqq C\|\Pi'_m u\|_s.$$

Before we do this, we will first show how Proposition 5.1 follows. By Lemma 3.1,

$$\Pi'_m=\Pi_m+g_{m-1}+g_{m-2}+\cdots,$$

where g_{m-j} satisfies condition (L). Hence,

$$P=\Pi'_m+(P'_{m-1}-g_{m-1})+(P'_{m-2}-g_{m-2})+\cdots,$$

where $P'_{m-i}-g_{m-i}$ satisfies condition (L). Thus (5.7) implies that

$$(5.8)\qquad \begin{aligned}&\sum_{j=1}^{r}\left(\frac{1}{T}\right)^{j}\left\|D^{t_j}\prod_{i=1}^{p}\partial_i^{[r_i-j]}u\right\|_s\\ &\leqq C\|Pu\|_s+C\sum_{i=1}^{m}\|(P'_{m-i}-g_{m-i})u\|_s.\end{aligned}$$

Since by condition (L), $P'_{m-i}-g_{m-i}$ is of the same form as the operator in the l.h.s. of (5.8), we can absorb the second term of the r.h.s. of (5.8) into the l.h.s. and get

$$(5.9)\qquad \sum_{j=1}^{r}\left(\frac{1}{T}\right)^{j}\left\|D^{t_j}\prod_{i=1}^{p}\partial_i^{[r_i-j]}u\right\|_s \leqq C\|Pu\|_s.$$

When $j=r$, $D^{t_j}\prod_{i=1}^{p}\partial_i^{[r_i-j]}u=D^{m-r}u$, and (5.9) yields

$$\left(\frac{1}{T}\right)^{r}\|D^{m-r}u\|_s\leqq C\|Pu\|_s.$$

So, what is left is to show how (5.7) follows from (5.6). To prove (5.7) it suffices to show that

$$(5.10)_l\qquad \sum_{j=l+1}^{r}\left(\frac{1}{T}\right)^{j}\left\|D^{t_j}\prod_{i=1}^{p}\partial_i^{[r_i-j]}u\right\|_s\leqq C\left(\frac{1}{T}\right)^{l}\left\|D^{t_l}\prod_{i=l+1}^{r}P_{m_i}u\right\|_s$$

for $0\leqq l\leqq r-1$.

We prove $(5.10)_l$ by a backwards induction on l. First, the same argument used to prove (5.6) shows that

$$\sum_{j=l+1}^{r}\left(\frac{1}{T}\right)^{j}\left\|D^{t_j}\prod_{i=j+1}^{r}P_{m_i}u\right\|_s \leqq C\left(\frac{1}{T}\right)^{l}\left\|D^{t_l}\prod_{i=l+1}^{r}P_{m_i}u\right\|_s, \tag{5.11}$$

for each l, $0\leqq l\leqq r-1$. If $l=r-1$, (5.11) becomes

$$\left(\frac{1}{T}\right)^{r}\|D^{m-r}u\|_s \leqq C\left(\frac{1}{T}\right)^{r-1}\|D^{t_{r-1}}P_{m_r}u\|_s,$$

which is $(5.10)_{r-1}$. Now suppose $(5.10)_l$ is true; we will show that $(5.10)_{l-1}$ is also true. (5.11) implies that

$$\left(\frac{1}{T}\right)^{l}\left\|D^{t_l}\prod_{i=l+1}^{r}P_{m_i}u\right\|_s \leqq C\left(\frac{1}{T}\right)^{l-1}\left\|D^{t_{l-1}}\prod_{i=l}^{r}P_{m_i}u\right\|_s. \tag{5.12}$$

It can be seen easily that we can rearrange the ∂_i in $\prod_{i=l+1}^{r}P_{m_i}$ to get, modulo lower order terms, $\prod_{i=1}^{p}\partial_i^{[r_i-l]}$. Thus,

$$\left\|D^{t_l}\prod_{i=1}^{p}\partial_i^{[r_i-l]}u\right\|_s = \left\|D^{t_l}\prod_{i=l+1}^{r}P_{m_i}u + M_l u\right\|_s,$$

where $M_l u$ are the lower order terms which arise from the rearrangement. Hence

$$\left(\frac{1}{T}\right)^{l}\left\|D^{t_l}\prod_{i=1}^{p}\partial_i^{[r_i-l]}u\right\|_s \leqq \left(\frac{1}{T}\right)^{l}\left\|D^{t_l}\prod_{i=l+1}^{r}P_{m_i}u\right\|_s + \left(\frac{1}{T}\right)^{l}\|M_l u\|_s. \tag{5.13}$$

By the induction hypothesis, we have

$$\sum_{j=l+1}^{r}\left(\frac{1}{T}\right)^{j}\left\|D^{t_j}\prod_{i=1}^{p}\partial_i^{[r_i-j]}u\right\|_s \leqq C\left(\frac{1}{T}\right)^{l}\left\|D^{t_l}\prod_{i=l+1}^{r}P_{m_i}u\right\|_s. \tag{5.14}$$

Adding (5.13) and (5.14) we then have

$$\sum_{j=l}^{r}\left(\frac{1}{T}\right)^{j}\left\|D^{t_j}\prod_{i=1}^{p}\partial_i^{[r_i-j]}u\right\|_s \leqq C\left(\frac{1}{T}\right)^{l}\left\|D^{t_l}\prod_{i=l+1}^{r}P_{m_i}u\right\|_s + C\left(\frac{1}{T}\right)^{l}\|M_l u\|_s. \tag{5.15}$$

Comparing this with (5.12), we see that

$$\sum_{j=l}^{r}\left(\frac{1}{T}\right)^{j}\left\|D^{t_j}\prod_{i=1}^{p}\partial_i^{[r_i-j]}u\right\|_s \leqq C\left(\frac{1}{T}\right)^{l}\left\|D^{t_l}\prod_{i=l+1}^{r}P_{m_i}u\right\|_s,$$

which is $(5.10)_{l-1}$. Hence $(5.10)_l$ is true for all l and the proof is complete.

REMARK. Examining estimate (5.9) more closely, we see that under condition (L),

$$\sum_{j=1}^{m}\left(\frac{1}{T}\right)^{j}\|P'_{m-j}u\|_s \leqq C\|\Pi_m u\|_s. \tag{5.16}$$

Now, (5.16) implies that

$$\sum_{j=1}^{m}\|P'_{m-j}u\|_s \leqq \varepsilon\|\Pi_m u\|_s, \tag{5.17}$$

where ε is as small as we like if we take T small enough. On the other hand, since (5.17) leads to the well-posedness of the Cauchy problem for operator P, once we show that condition (L) is necessary for the Cauchy problem to be well-posed, we will arrive at the conclusion that estimate (5.17) holds only under condition (L). We thus have another criterion for hyperbolicity, namely, that P is hyperbolic if and only if it is dominated by its "modified" principal part Π_m in the sense of (5.17). This generalizes the domination condition formulated by Hörmander for operators with constant coefficients. (See Hörmander [4], theorem 5.58.) Related results were presented by L. Svensson [16].

6. Necessity of condition (L)

Rather than proving directly that condition (L) is necessary for the Cauchy problem to be well-posed, we will show instead that condition (L) is equivalent to another condition originally formulated by H. Flaschka and G. Strang [2] which they showed to be necessary. In the process, we will formulate another condition which is a generalization of a condition proved by S. Mizohata and Y. Ohya [13], [14] to be both necessary and sufficient for hyperbolicity in the restricted case where the multiplicity of the characteristics is at most two. We will show the equivalence of all three conditions.

First we need some preliminary steps. Denote by $\tilde{\partial}_i$, $\tilde{\partial}_i = D_t - \tilde{\lambda}_i(x,t,D_x)$, $1 \leqq i \leqq m$, where $\tilde{\lambda}_i \in L^1_x$ is arbitrary. $\tilde{\partial}_0 \equiv I$. We have the following lemma, which shows how the $\tilde{\partial}_i$ can be utilized as directional derivatives.

LEMMA 6.1. (a). *For $j \geqq 0$ there exist $a_i(x,t,D_x) \in L^i_x$ such that*

$$\tilde{\partial}_j\tilde{\partial}_{j-1}\cdots\tilde{\partial}_1 = \sum_{i=0}^{j} a_i(x,t,D_x)D_t^{j-i} + T_1,$$

where T_1 represents the lower order terms; i.e. $T_1 = \sum_{i=0}^{j-1} c_i(x,t,D_x)D_t^{j-i-1}$, order $c_i \leqq i$.

(b) *Conversely, there exist* $b_i(x,t,D_x) \in L_x^i$ *such that*

$$D_t^j = \sum_{i=0}^{j} b_i(x,t,D_x)\,\tilde{\partial}_{j-i}\tilde{\partial}_{j-i-1}\cdots\tilde{\partial}_0 + T_2,$$

where $T_2 = \sum_{i=0}^{j-1} d_i(x,t,D_x)\,\tilde{\partial}_{j-i-1}\tilde{\partial}_{j-i-2}\cdots\tilde{\partial}_0$, *order* $d_i \leqq i$.

COROLLARY 6.2. *For every operator* $G \in L_{x,t}^k$, *there exists* $c_i(x,t,D_x) \in L_x^i$ *such that*

$$G = \sum_{i=0}^{k} c_i(x,t,D_x)\,\tilde{\partial}_{k-i}\tilde{\partial}_{k-i-1}\cdots\tilde{\partial}_0 + T,$$

for some operator $T \in L_{x,t}^{k-1}$. *(More precisely,* T *has the form* $T = \sum_{i=0}^{k-1} e_i(x,t,D_x)\tilde{\partial}_{k-i-1}\tilde{\partial}_{k-i-2}\cdots\partial_0$, *order* $e_i \leqq i$*).*

PROOF. Lemma 6.1 is proved easily by induction on j. Its proof as well as that of Corollary 6.2 are left to the reader.

We are now ready to state the condition which will generalize that formulated by Mizohata and Ohya. Recall that our operator P is of the form

$$P = \Pi_m + P'_{m-1} + P'_{m-2} + \cdots, \quad \text{where} \quad \Pi_m = \prod_{i=1}^{p} \partial_i^{r_i}$$

As in the proof of Proposition 3, we consider the operator

$$\Pi'_m = \left(\prod_{i=1}^{p} \partial_{p-i+1}\right)^{r_1}\left(\prod_{i=1}^{p-1} \partial_{p-i+1}\right)^{r_2-r_1}\cdots\left(\prod_{i=1}^{p-k} \partial_{p-i+1}\right)^{r_{k+1}-r_k}\cdots(\partial_p)^{r_p-r_{p-1}};$$

$\Pi'_m = P_{m_1}P_{m_2}\cdots P_{m_r}$, where each P_{m_i} is of the form $\Pi\,\partial_i$ of order m_i, $r = \max r_i = r_p$. Note that in particular $m_1 = p$.

We now introduce the following operators which will serve as a basis for the lower order terms we allow:

$$\Delta_0^k = 1, \quad \Delta_1^k = \partial_p, \quad \Delta_2^k = \partial_p\partial_{p-1}, \quad \cdots,$$

$$\Delta_{m_{k+1}}^k = \partial_p\partial_{p-1}\cdots\partial_{p-m_{k+1}+1} = P_{m_{k+1}}, \quad \Delta_{m_{k+1}+1}^k = P_{m_{k+1}}\partial_p,$$

$$\Delta_{m_{k+1}+2}^k = P_{m_{k+1}}\partial_p\partial_{p-1}, \quad \cdots, \quad \Delta_{m_{k+1}+m_{k+2}}^k = P_{m_{k+1}}P_{m_{k+2}}, \quad \cdots,$$

$$\Delta_{m_{k+1}+m_{k+2}+\cdots+m_r}^k = P_{m_{k+1}}P_{m_{k+2}}\cdots P_{m_r}, \quad \cdots,$$

$$\Delta_{m_{k+1}+m_{k+2}+\cdots+m_r+m_k}^k = P_{m_{k+1}}P_{m_{k+2}}\cdots P_{m_r}P_{m_k}, \cdots,$$

$$\Delta_{m-m_1}^k = \Delta_{m-p}^k = \Delta_{m_{k+1}+\cdots+m_r+m_k+m_{k-1}+\cdots+m_2}^k = P_{m_{k+1}}\cdots P_{m_r}\cdots P_{m_2},$$

$$\Delta_{m-p+1}^k = \Delta_{m-m_1}^k\partial_p, \quad \cdots, \quad \Delta_{m-j}^k = \Delta_{m-m_1}^k\partial_p\partial_{p-1}\cdots\partial_{p-j+1}.$$

By Corollary 6.2, there exist operators $C^k_{m-k-j} \in L_x^{m-k-j}$ such that

$$P'_{m-k}(x,t,D_x,D_t) = \sum_{j_k=0}^{m-k} C^k_{m-k-j_k}(x,t,D_x)\Delta^k_{j_k}, \quad k = 1,2,\cdots,r-1.$$

We are now ready to formulate the condition:

Condition (M)

$$C^k_{m-k-j_k} \equiv 0, \qquad \begin{aligned} &k = 1,\cdots,r-1; \\ &j_k = 0,1,\cdots,m_{k+1}+\cdots+m_r-1. \end{aligned}$$

Example. If the multiplicity of the characteristics is at most two, $r = 2$. Hence under condition (M) we need only put a restriction on P'_{m-1}. Now suppose $\Pi_m = \Pi_{j=s+1}^{m-s} \partial_i \Pi_{j=1}^s \partial_i^2$, as in the case considered by Mizohata and Ohya [13], [14]. Then by our notation,

$$\Pi'_m = (\partial_s\partial_{s-1}\cdots\partial_1\partial_{m-s}\partial_{m-s-1}\cdots\partial_{s+1})(\partial_s\partial_{s-1}\cdots\partial_1) = P_{m_1}P_{m_2},$$

where $m_1 = m-s$ and $m_2 = s$. The basis for P'_{m-1} is:

$$\Delta_0^1 = 1, \quad \Delta_1^1 = \partial_s, \quad \partial_2^1 = \partial_s\partial_{s-1}, \quad \cdots, \quad \Delta_s^1 = \partial_s\cdots\partial_1,$$

$$\Delta_{s+1}^1 = \partial_s\partial_{s-1}\cdots\partial_1\partial_s, \quad \cdots, \quad \Delta_{m-1}^1 = \partial_s\cdots\partial_1\partial_s\cdots\partial_1\partial_{m-s}\cdots\partial_{s+2}.$$

Then by Corollary 6.2, there exist $C^1_{m-j-1} \in L_x^{m-j-1}$ such that

$$P'_{m-1}(x,t,D_x,D_t) = \sum_{j=0}^{m-1} C^1_{m-j-1}(x,T,D_x)\Delta^1_j.$$

Condition (M) then becomes

$$C^1_{m-j-1}(x,t,D_x) \equiv 0, \qquad j = 0,\cdots,s-1. \tag{6.1}$$

Since in this case there is no condition on the terms of order lower than $m-1$, (6.1) can be weakened to $C^1_{m-j-1}(x,t,\xi) \equiv 0$ for $j = 0,\cdots,s-1$. This is precisely the condition formulated by Mizohata and Ohya.

Finally, let us recount the condition first presented by H. Flaschka and G. Strang [2], which they proved to be necessary for the Cauchy problem to be well-posed:

Condition (F). If φ is characteristic with respect to a root λ_i of multiplicity N, i.e. if $\varphi_t = \lambda_i(x,t,\text{grad}_x\varphi)$, and if $f(x,t) \in C^\infty$, then

$$P(fe^{i\rho\varphi}) = O(\rho^{m-N}), \quad \text{as} \quad \rho \to \infty.$$

We then have the following:

THEOREM 4. *The following are equivalent*:
(1) *Condition* (M),
(2) *Condition* (L),
(3) *Condition* (F).

COROLLARY 6.3. *Condition* (L) *is necessary for the Cauchy problem to be well-posed.*

REMARK 1. Combining Proposition 3 and Corollary 6.3, the proof of Theorem 2 is completed.

REMARK 2. Theorem 4, besides showing the necessity of condition (L) for hyperbolicity, also proves that condition (F) is sufficient. This has been proved earlier in a more direct manner by J. Chazarain [1]. In addition, it proves both the sufficiency and necessity of condition (M).

PROOF OF THEOREM 4. Proving that (1) $\Rightarrow$ (2) is easy. This is because each term

$$\Delta^k_{j_k}, \qquad \begin{aligned} &k = 1, 2, \cdots, r-1; \\ &j_k = m_{k+1} + \cdots + m_r, \cdots, m-k \end{aligned}$$

after a permutation of the ∂_i in the product, if necessary, satisfies condition (L), modulo lower order terms. As for the lower order terms arising from the commutation of the ∂_i, they too satisfy condition (L) because of Lemma 3.1.

(2) $\Rightarrow$ (3) is also straightforward since condition (L) is sufficient for hyperbolicity while condition (F) is necessary.

What remains is to show that (3) $\Rightarrow$ (1). Our proof will follow that given by Flaschka and Strang in the special case $r = 2$.

We will compare the two conditions in a particularly suitable coordinate system. In order to accomplish this, we will need some lemmas first presented by Flaschka and Strang [2]. We refer the reader to their paper for the proofs.

LEMMA 6.4. *Let $\varphi(x, t)$ be the characteristic for the root $\lambda^*(x, t, \xi)$ of multiplicity N of $P_m(x, t, \xi, \tau)$ and suppose $D_{x_1}\varphi(x, t) \neq 0$. Under the change of variables*

$$\begin{aligned} &x'_1 = \varphi(x, t), \\ &x'_j = x_j, \qquad j \geqq 2, \\ &t' = t, \end{aligned} \tag{6.2}$$

an operator P satisfying condition (F) *will transform into an operator*

$$P' = P'(x', t', D_{x'}, D_{t'}) = \sum_{|\alpha| \leq m} a'_\alpha(x', t') D'^\alpha$$

in which $a'_\alpha \equiv 0$ *if* $\alpha > m - N$.

LEMMA 6.5. *Condition* (M) *is invariant under the change of coordinates* (6.2).

LEMMA 6.6. *Subjected to the transformation* (6.2), *a characteristic root* $\lambda(x, t, \xi)$ *of* P *goes over into a characteristic root* $\lambda'(x', t', \xi')$ *of* P'. *The transform* $\lambda^{*\prime}$ *of the root* λ^* *distinguished in Lemma* 6.4 *has a finite limit*

$$\lim \lambda^*(x', t', \xi'), \xi'_1 \to \infty;$$

for any other root λ', *the limit*

$$\lim \frac{\lambda'(x', t', \xi')}{\xi'_1}, \qquad \xi'_1 \to \infty$$

is finite.

We now suppose that while

$$C^k_{m-k-j_k}(x, t, D_x) \equiv 0, \qquad \begin{array}{l} k = 1, \cdots, \bar{k} - 1; \\ j_k = 0, \cdots, m_{k+1} + \cdots m_r - 1 \end{array}$$

and

$$C^{\bar{k}}_{m-\bar{k}-j_{\bar{k}}}(x, t, D_x) \equiv 0, \qquad j_{\bar{k}} = 0, \cdots, l - 1,$$

there does exist a term with

$$C^{\bar{k}}_{m-\bar{k}-l}(0, 0, \xi^0) \neq 0, \quad \text{for some fixed } \xi^0 \neq 0,$$
$$\text{and some } l < m_{\bar{k}+1} + \cdots + m_r - 1.$$

This violates condition (M). Rotating coordinates, if necessary, we may assume that $\xi^0_1 \neq 0$. In order to prove that condition (F) is also violated, we need only exhibit a characteristic function φ for which the associated transformation (6.2) yields an operator P' which dosen't have the property required by Lemma 6.4. Since $P_{m_r} = \Pi_{i=1}^{p-s+1} \partial_{p-i+1}$ (see Section 5) we choose φ to be characteristic for the root λ_s, with initial values $\varphi(x, 0) = x \cdot \xi^0$. The transformation (6.2) then changes $C^{\bar{k}}_{m-\bar{k}-l}(x, t, \xi)$ into $C'^{\bar{k}}_{m-\bar{k}-l}(x', t', \xi')$, a homogeneous function of ξ' with the property

$$C'^{\bar{k}}_{m-\bar{k}-l}(x', 0, \xi'_1, 0, \cdots, 0) = |\xi'_1|^{m-\bar{k}-l} C^{\bar{k}}_{m-\bar{k}-l}(x, 0, \xi^0) \tag{6.4}$$

which does not vanish for x close to 0.

Next, $\Delta^{\bar{k}}_l(x, t, D_x, D_t)$ will transform into an operator with principal symbol

$$\Delta'^{\bar{k}}_l(x', t', \xi', \tau') = \prod_{i=1}^{p} (\tau' - \lambda_i(x', t', \xi'))^{t_i}, \tag{6.5}$$

for $t_i < r_i - \bar{k}$.

Thus $C^{\bar{k}}_{m-\bar{k}-l}(x, t, D_x)\, \Delta^{\bar{k}}_l(x, t, D_x, D_t)$ is changed into an operator with $m - \bar{k}$th order symbol

$$C'^{\bar{k}}_{m-k-l}(x', t', \xi')\Delta^{\bar{k}}_l(x', t', \xi', \tau'). \tag{6.6}$$

Near $x' = 0$, $t' = 0$, by (6.4), $C'^{\bar{k}}_{m-\bar{k}-l}(x', t', \xi')$, grows as fast as $(\xi'_1)^{m-\bar{k}-l}$. As for $\Delta'^{\bar{k}}_l(x', t', \xi', \tau')$, since t_s is less than $r_s - \bar{k} = r - \bar{k}$, we see, after applying Lemma 6.6 to (6.5), that $\Delta'^{\bar{k}}_l$ grows as fast as $(\xi'_1)^{l-t_s}$. Thus (6.6) grows as fast as $(\xi'_1)^{m-\bar{k}-l+l-t_s}$, where $m - \bar{k} - l + l - t_s = m - \bar{k} - t_s > m - k - (r - \bar{k}) = m - r$. This surpasses the growth $O(\xi'^{m-r}_1)$ permitted by Lemma 6.4. It can be seen easily that the $m - \bar{k}$th order terms of Π_m, as well as the $m - k$th contribution from P'_{m-k}, $k > \bar{k}$ (these terms arise since neither Π_m nor P'_{m-k} need be homogeneous) cannot cancel the influence of term (6.6). (The basic step is seeing that the terms Π_m and P'_{m-k}, $k > \bar{k}$, all have at most the growth $O(\xi'^{m-r}_1)$.) This proves that (3) $\Rightarrow$ (1) and completes the proof of Theorem 4.

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